2014

PHYSICS

(Major)

Paper : 3.1

Full Marks : 60

Time : 2½ hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Mathematical Physics)

(*Marks* : 25)

1. Answer the following questions : 1×3=3

(*a*) Define self-adjoint matrix.

(*b*) Show that trace of the sum of two matrices is the sum of their traces.

(*c*) Find the conjugate transpose of the following matrix :

$$A = \begin{pmatrix} 2+3i & -1+2i \\ i & 5-6i \end{pmatrix}$$

2. Show that the matrix A given by

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

is unitary. 2

3. Answer any *two* of the following questions :

(a) (i) If A and B are two Hermitian matrices, then prove that AB is Hermitian only if A and B commute. 1

(ii) Solve the following system of equations by the use of matrix method : 2

$$x + 3y = 4$$
$$2x - 2y = 6$$

(iii) If

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 4 & 1 & 5 \\ -3 & 2 & 4 \end{pmatrix}$$

then find B, when $A^T + 2B = 3I$. 2

(b) (i) Show that

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

is a nilpotent matrix of index 2. 2

(ii) Prove that the modulus of each eigenvalue of an orthogonal matrix is unity. 3

(c) A reference frame a rotates with respect to another reference frame b with uniform angular velocity $\vec{\omega}$. If the position, velocity and acceleration of a particle in frame a are represented by R, v_a and f_a respectively, then show that the acceleration of that particle in frame b is given by f_b, where

$$f_b = f_a + 2\vec{\omega} \times v_a + \vec{\omega} \times (\vec{\omega} \times R)$$

How will the expression get modified if the frame a rotates with respect to frame b with non-uniform angular velocity $\vec{\omega}$?

4+1=5

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) (i) Verify the theorem

$$A(\text{adj } A) = (\text{adj } A)A = |A|I$$

using

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$$

3½

(ii) For the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

show that

$$[\sigma_1, \sigma_2] = 2i\sigma_3$$ 1½

(b) (i) Express the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & -2 \\ 4 & 2 & 0 \end{bmatrix}$$

as the sum of a symmetric and a skew-symmetric matrix. 2

(ii) Let the matrix

$$[A] = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$$

is transformed to the diagonal form

$$[B] = T_\theta A T_\theta^{-1}$$

where

$$T_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Show that

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$ 3

(c) (i) What is a special square matrix? 1

(ii) By using the Cayley-Hamilton theorem, compute the inverse of

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$ 4

(d) Diagonalize the following matrix : 5

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

GROUP—B

(Electrostatics)

(*Marks* : 35)

5. Choose the correct option : 1×4=4

(a) The energy density in an electrostatic field is

(i) $\frac{E^2}{2\varepsilon}$

(ii) $\frac{\varepsilon E^2}{2}$

(iii) $\frac{2\varepsilon}{E^2}$

(b) The permittivity of a medium has the unit

(i) $\frac{F}{m}$

(ii) $F \cdot m$

(iii) $\frac{N}{m}$

(c) The electric field $\vec{E}$ and the electric potential ϕ are related by

(i) $\vec{E} = \vec{\nabla}\phi$

(ii) $\vec{E} = -\vec{\nabla}\phi$

(iii) $\phi = \vec{\nabla} \cdot \vec{E}$

(d) The dielectric constant K and the electrical susceptibility χ of a dielectric material are related by

(i) $K = 1 + \chi$

(ii) $\chi = 1 + K$

(iii) $K\chi = 1$

6. Answer the following questions : 3×2=6

(a) Check whether the following functions may be possible for electrostatic fields

$$\vec{E} = (2x\hat{i} - yz^2\hat{j} - \hat{k} - y^2z\hat{k})A$$

where A is a constant with suitable dimensions. Using Poisson's equation, find how the charge density changes with position.

Or

Find $\vec{\mathbb{E}}$ at $(0, 0, 5)\,m$ due to $Q_1 = 5\,\mu C$ at $(0, 3, 0)\,m$ and $Q_2 = 5\mu C$ at $(3, 0, 0)\,m$.

(b) If ρ' be the density of polarization charges within the volume of a dielectric slab placed in an electric field, then prove that $\rho' = -\vec{\nabla}\cdot\vec{P}$.

7. Using integral form of Gauss law in electrostatics, determine the electric field and potential at a distance r from a straight infinitely long wire having a charge λ per unit length. 2½+2½=5

Or

What is electric dipole? Show that the electric field in free space due to a dipole is given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\left[\frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^2} - \vec{p}\right]$$

where $\vec{p}$ is the dipole moment. 5

8. Answer any *two* questions :

(*a*) (*i*) Find an expression for the torque experienced by an electric dipole in external electric field. Hence show that the work done in rotating the dipole from an initial position θ_1 to the final position θ_2 is

$$W = -pE(\cos\theta_2 - \cos\theta_1)$$ 3+2=5

(*ii*) Find an expression for the potential energy due to the mutual interaction between two dipoles of dipole moments $\vec{p}_1$ and $\vec{p}_2$ respectively. Two water molecules each having a dipole moment $6 \cdot 2 \times 10^{-30}$ coulomb-metre point in the same direction and are inclined at an angle of 60° to the line joining their centres. Determine the potential energy due to their dipole-dipole interactions when their centres are $3 \cdot 1 \times 10^{-10}$ metre apart. 3+2=5

(b) (i) A uniformly charged sphere of radius a carries a total charge Q and a volume density of charge ρ. Show that the electrostatic energy is

$$U = \frac{3Q^2}{20\pi\varepsilon_0 a}$$ 3

(ii) State and prove uniqueness theorem. 4

(iii) Show that the potential inside a spherical capacitor is given by

$$V = 4\pi\varepsilon_0 \left(\frac{ab}{b-a}\right)$$

where a and b are the radii of the inner and outer concentric spheres respectively. 3

(c) (i) Write Poisson's equation. Solve Laplace's equation to find the potential at a distance $\vec{r}$ from the axis of an infinitely long conducting cylinder of radius a_0 charged with a surface density σ. Take the potential of the cylinder to be zero. 1+4=5

(ii) Calculate with the method of electrical image the potential and the field at any point in space when a point charge is placed in front of a conducting plane of infinite extent maintained at zero potential. 5

(d) (i) Define electrical susceptibility. 1

(ii) An isotropic dielectric is placed in an otherwise uniform electrostatic field $\vec{E}$. Show that field inside a spherical cavity in this direction is

$$\vec{E}_i = \vec{E} + \frac{\vec{P}}{3\varepsilon_0}$$

where $\vec{P}$ is the polarization. 4

(iii) Establish the Clausius-Mosotti equation

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha}{3\varepsilon_0}$$

for a linear dielectric material. 5

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